4.4 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

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What You Should Learn

- Evaluate trigonometric functions of any angle.
- Find reference angles.
- Evaluate trigonometric functions of real numbers.

Introduction

Introduction

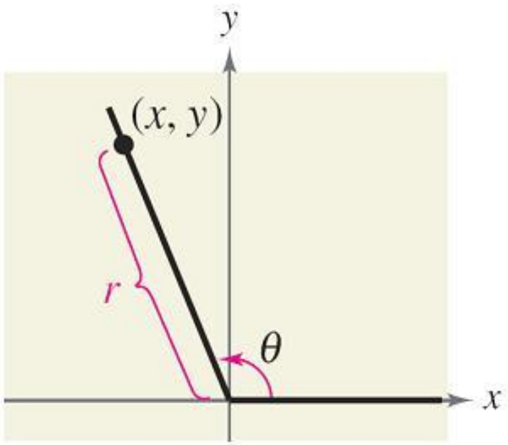
Definitions of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$



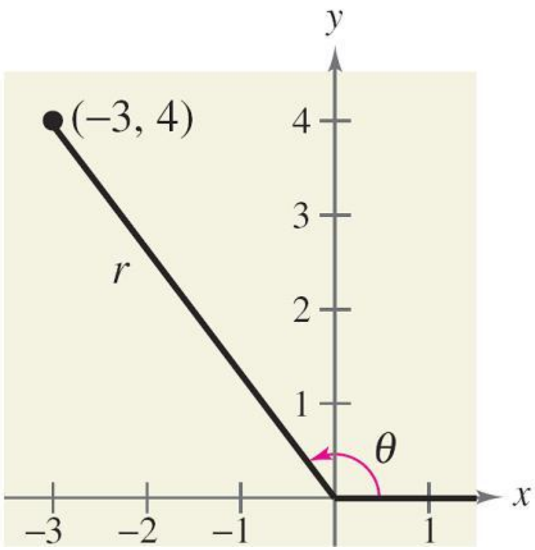
Example 1 – *Evaluating Trigonometric Functions*

Let (–3, 4) be a point on the terminal side of θ. Find the sine, cosine, and tangent of θ.

Solution:

$x = -3$, $y = 4$,

$$r = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{25}$$
$$= 5.$$



Example 1 – *Solution*

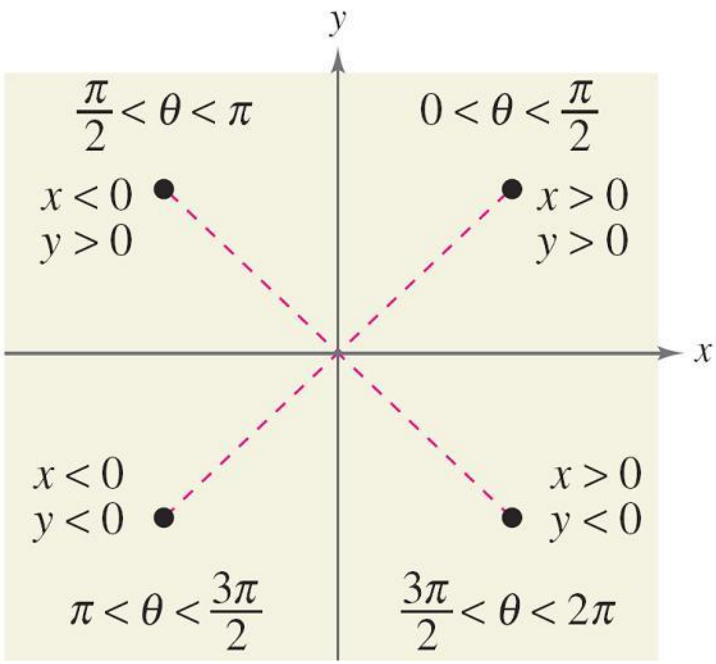
cont'd

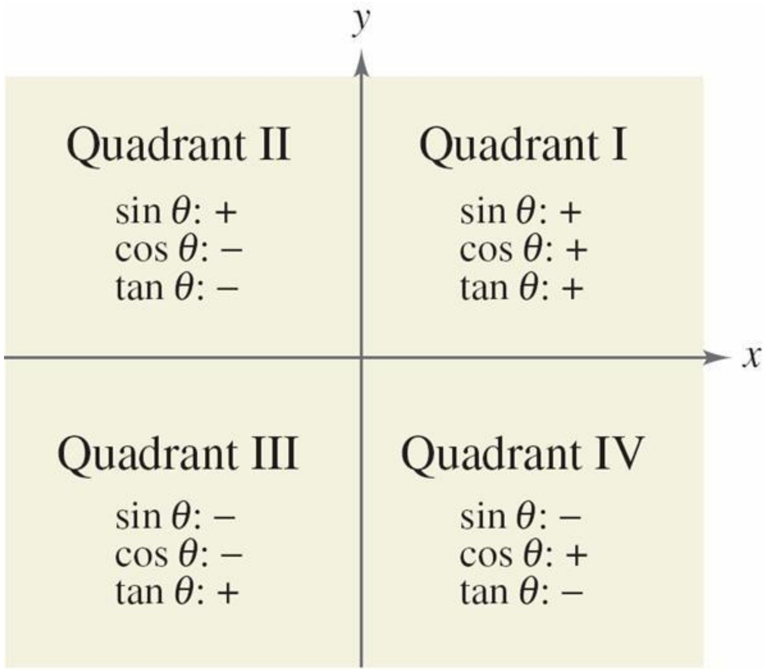
$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\cos\theta = \frac{x}{r} = -\frac{3}{5}$$

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}$$

Introduction





Reference Angles

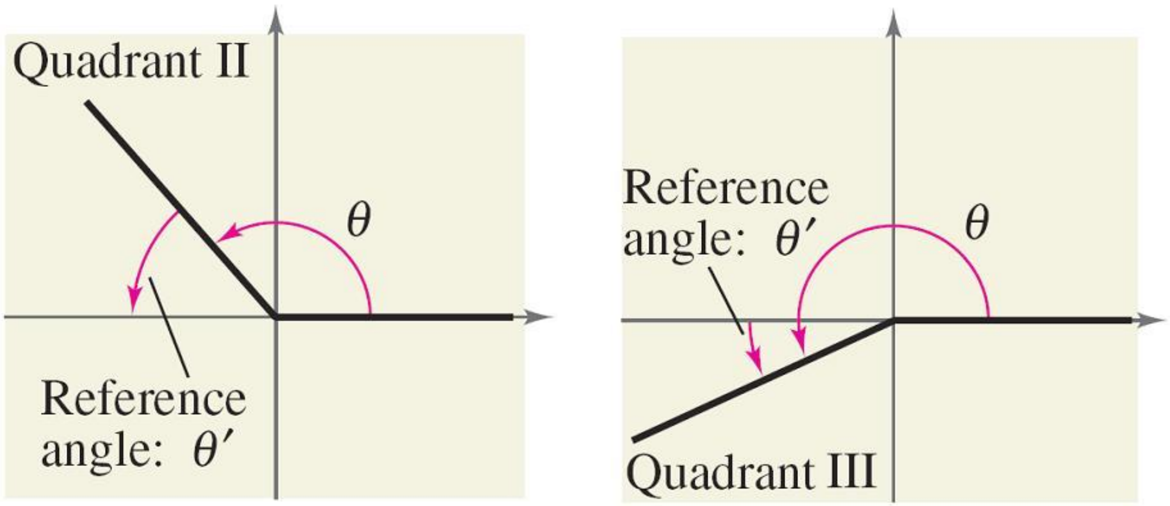
Reference Angles

Definition of Reference Angle

Let θ be an angle in standard position. Its **reference angle** is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

Reference Angles

The reference angles for θ in Quadrants II, III, and IV.



$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

Example 4 – *Finding Reference Angles*

Find the reference angle θ'.

a. $\theta = 300°$

b. $\theta = 2.3$

c. $\theta = –135°$

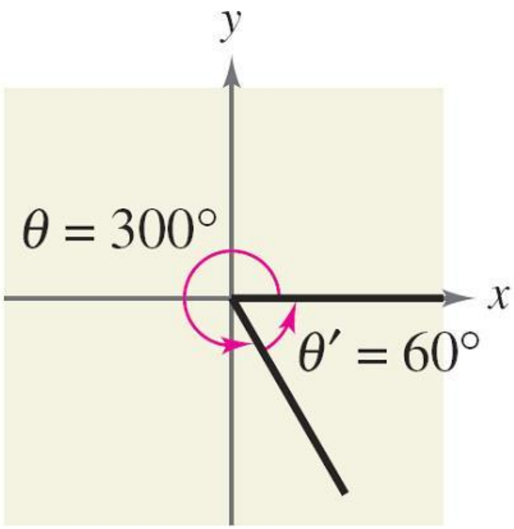
Example 4(a) – *Solution*

Because 300° lies in Quadrant IV, the angle it makes with the *x*-axis is

$$\theta' = 360° - 300°$$

$$= 60°.$$

Degrees



Example 4(b) – *Solution*

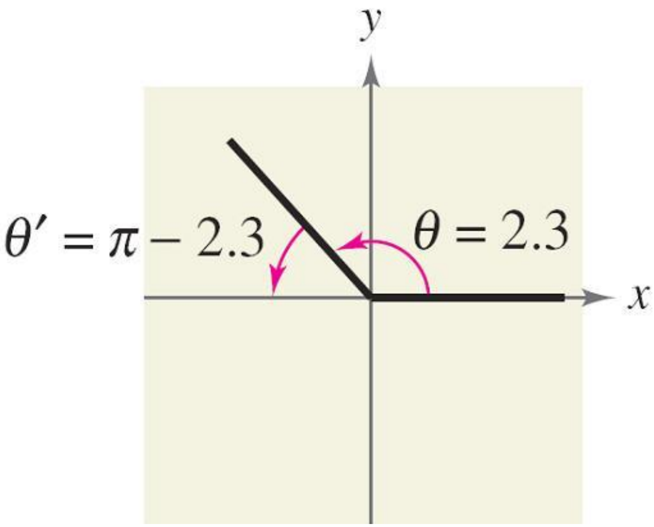
cont'd

Because 2.3 lies between $\pi/2 \approx 1.5708$ and $\pi \approx 3.1416$, it follows that it is in Quadrant II and its reference angle is

$$\theta' = \pi - 2.3$$

$$\approx 0.8416.$$

Radians



Example 4(c) – *Solution*

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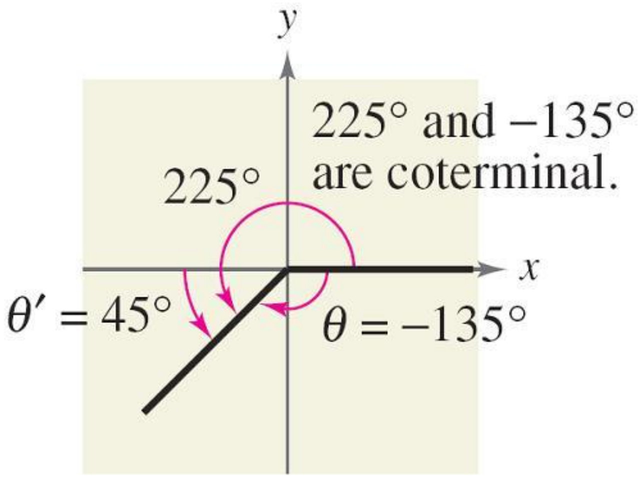
First, determine that –135° is coterminal with 225°, which lies in Quadrant III. So, the reference angle is

$$\theta' = 225° - 180°$$

$$= 45°.$$

Degrees



Trigonometric Functions of Real Numbers

Trigonometric Functions of Real Numbers

By definition, you know that

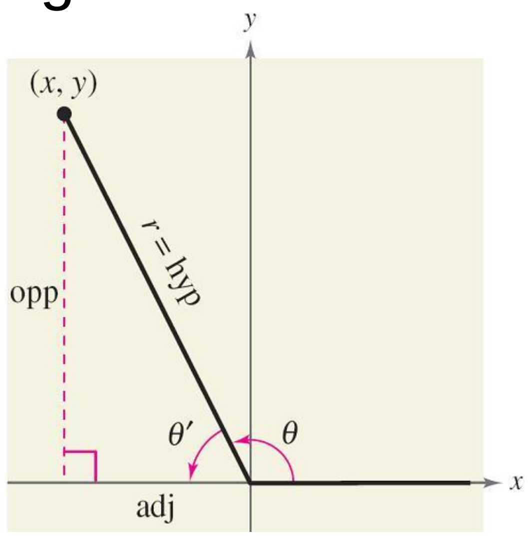
$$\sin\theta = \frac{y}{r} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}.$$

For the right triangle with acute angle θ' and sides of lengths $|x|$ and $|y|$, you have

$$\sin\theta' = \frac{\text{opp}}{\text{hyp}} = \frac{|y|}{r}$$

and

$$\tan\theta' = \frac{\text{opp}}{\text{adj}} = \frac{|y|}{|x|}.$$



opp = |y|, adj = |x|

Trigonometric Functions of Real Numbers

Evaluating Trigonometric Functions of Any Angle

To find the value of a trigonometric function of any angle θ:

1. Determine the function value for the associated reference angle θ'.
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undef.	0	Undef.

Example 5 – *Using Reference Angles*

Evaluate each trigonometric function.

a. cos $\frac{4\pi}{3}$

b. tan(–210°)

c. csc $\frac{11\pi}{4}$

Example 5(a) – *Solution*

cont'd

Because $\theta = 4\pi/3$ lies in Quadrant III, the reference angle is

$$\theta' = \frac{4\pi}{3} - \pi$$

$$= \frac{\pi}{3}$$

as shown in Figure 4.44.

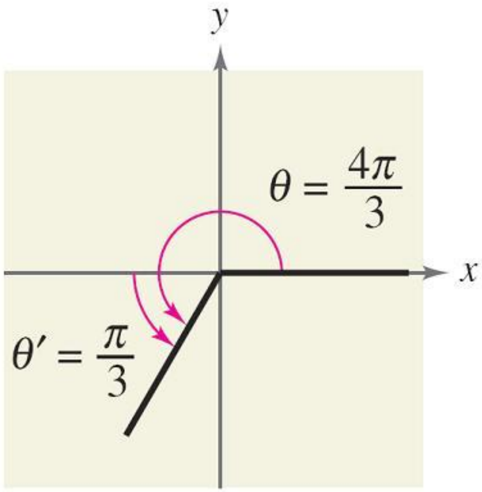


Figure 4.44

Moreover, the cosine is negative in Quadrant III, so

$$\cos\frac{4\pi}{3} = (-)\cos\frac{\pi}{3}$$

$$= -\frac{1}{2}.$$

Example 5(b) – *Solution*

cont'd

Because –210° + 360° = 150°, it follows that –210° is coterminal with the second-quadrant angle 150°.

So, the reference angle is θ' = 180° – 150° = 30°, as shown in Figure 4.45.

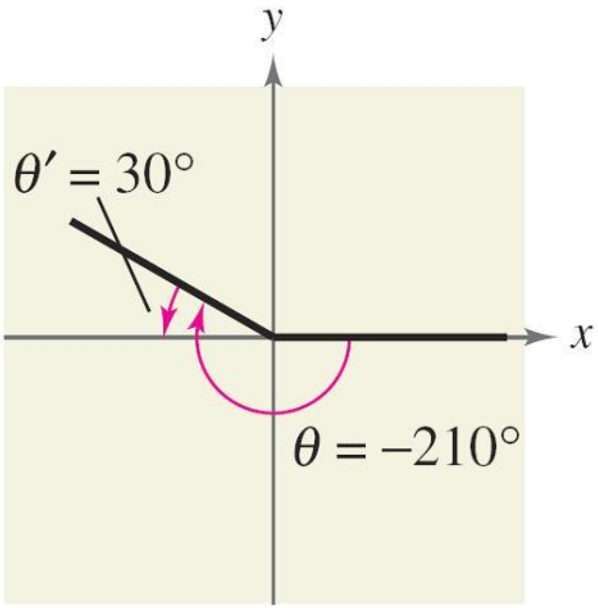


Figure 4.45

Example 5(b) – *Solution*

cont'd

Finally, because the tangent is negative in Quadrant II, you have

$$\tan(-210°) = (-)\tan 30°$$

$$= -\frac{\sqrt{3}}{3}.$$

Example 5(c) – *Solution*

cont'd

Because $(11\pi/4) - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$.

So, the reference angle is $\theta' = \pi - (3\pi/4) = \pi/4$, as shown in Figure 4.46.

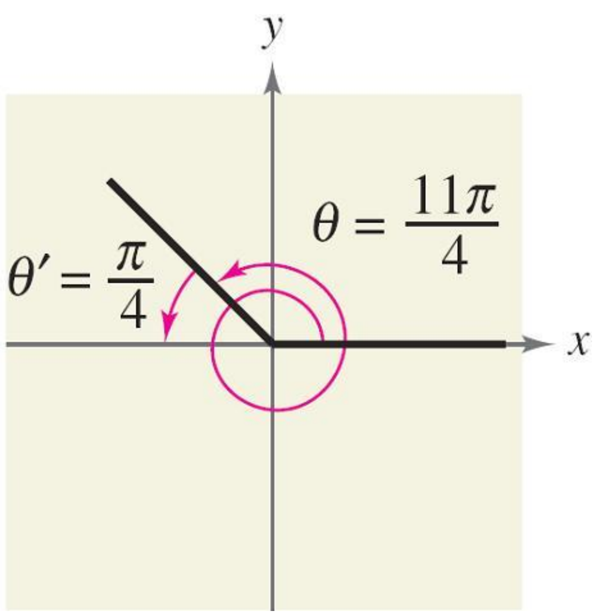


Figure 4.46

Example 5(c) – *Solution*

cont'd

Because the cosecant is positive in Quadrant II, you have

$$\csc\frac{11\pi}{4} = (+)\csc\frac{\pi}{4}$$

$$= \frac{1}{\sin(\pi/4)}$$

$$= \sqrt{2}.$$