

3.6 Quadratic Inequalities



Learning Standards
HSA-CED.A.1
HSA-CED.A.3

USING TOOLS STRATEGICALLY

To be proficient in math, you need to use technological tools to explore your understanding of concepts.

Essential Question How can you solve a quadratic inequality?

EXPLORATION 1 Solving a Quadratic Inequality

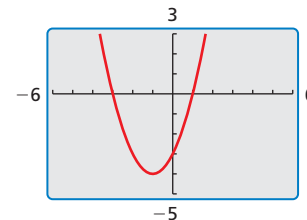
Work with a partner. The graphing calculator screen shows the graph of

$$f(x) = x^2 + 2x - 3.$$

Explain how you can use the graph to solve the inequality

$$x^2 + 2x - 3 \leq 0.$$

Then solve the inequality.



EXPLORATION 2 Solving Quadratic Inequalities

Work with a partner. Match each inequality with the graph of its related quadratic function. Then use the graph to solve the inequality.

a. $x^2 - 3x + 2 > 0$

b. $x^2 - 4x + 3 \leq 0$

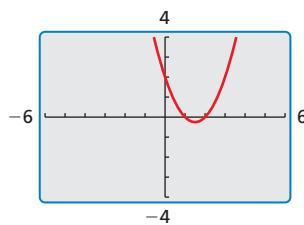
c. $x^2 - 2x - 3 < 0$

d. $x^2 + x - 2 \geq 0$

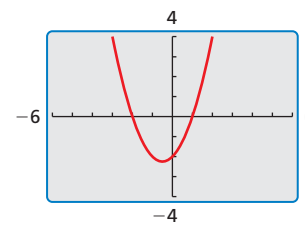
e. $x^2 - x - 2 < 0$

f. $x^2 - 4 > 0$

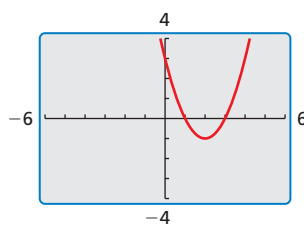
A.



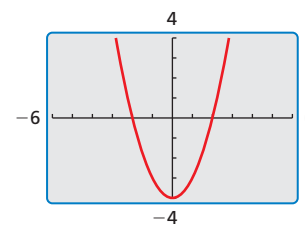
B.



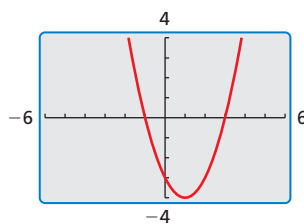
C.



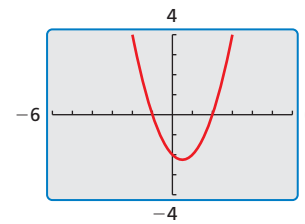
D.



E.



F.



Communicate Your Answer

3. How can you solve a quadratic inequality?

4. Explain how you can use the graph in Exploration 1 to solve each inequality. Then solve each inequality.

a. $x^2 + 2x - 3 > 0$

b. $x^2 + 2x - 3 < 0$

c. $x^2 + 2x - 3 \geq 0$

3.6 Lesson

Core Vocabulary

quadratic inequality in two variables, p. 140
quadratic inequality in one variable, p. 142

Previous

linear inequality in two variables

What You Will Learn

- ▶ Graph quadratic inequalities in two variables.
- ▶ Solve quadratic inequalities in one variable.

Graphing Quadratic Inequalities in Two Variables

A **quadratic inequality in two variables** can be written in one of the following forms, where a , b , and c are real numbers and $a \neq 0$.

$$y < ax^2 + bx + c \qquad y > ax^2 + bx + c$$

$$y \leq ax^2 + bx + c \qquad y \geq ax^2 + bx + c$$

The graph of any such inequality consists of all solutions (x, y) of the inequality.

Previously, you graphed linear inequalities in two variables. You can use a similar procedure to graph quadratic inequalities in two variables.

Core Concept

Graphing a Quadratic Inequality in Two Variables

To graph a quadratic inequality in one of the forms above, follow these steps.

- Step 1** Graph the parabola with the equation $y = ax^2 + bx + c$. Make the parabola *dashed* for inequalities with $<$ or $>$ and *solid* for inequalities with \leq or \geq .
- Step 2** Test a point (x, y) inside the parabola to determine whether the point is a solution of the inequality.
- Step 3** Shade the region inside the parabola if the point from Step 2 is a solution. Shade the region outside the parabola if it is not a solution.

EXAMPLE 1 Graphing a Quadratic Inequality in Two Variables

Graph $y < -x^2 - 2x - 1$.

SOLUTION

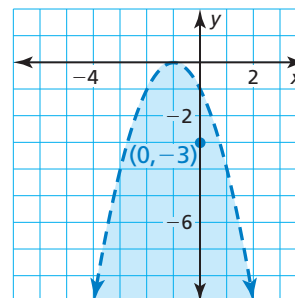
Step 1 Graph $y = -x^2 - 2x - 1$. Because the inequality symbol is $<$, make the parabola dashed.

Step 2 Test a point inside the parabola, such as $(0, -3)$.

$$\begin{aligned} y &< -x^2 - 2x - 1 \\ -3 &\stackrel{?}{<} -0^2 - 2(0) - 1 \\ -3 &< -1 \quad \checkmark \end{aligned}$$

So, $(0, -3)$ is a solution of the inequality.

Step 3 Shade the region inside the parabola.



LOOKING FOR STRUCTURE

Notice that testing a point is less complicated when the x -value is 0 (the point is on the y -axis).



EXAMPLE 2 Using a Quadratic Inequality in Real Life

A manila rope used for rappelling down a cliff can safely support a weight W (in pounds) provided

$$W \leq 1480d^2$$

where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

SOLUTION

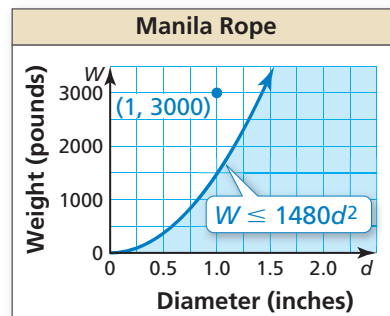
Graph $W = 1480d^2$ for nonnegative values of d . Because the inequality symbol is \leq , make the parabola solid. Test a point inside the parabola, such as $(1, 3000)$.

$$W \leq 1480d^2$$

$$3000 \stackrel{?}{\leq} 1480(1)^2$$

$$3000 \not\leq 1480$$

- Because $(1, 3000)$ is not a solution, shade the region outside the parabola. The shaded region represents weights that can be supported by ropes with various diameters.



Graphing a *system* of quadratic inequalities is similar to graphing a system of linear inequalities. First graph each inequality in the system. Then identify the region in the coordinate plane common to all of the graphs. This region is called the *graph of the system*.

EXAMPLE 3 Graphing a System of Quadratic Inequalities

Graph the system of quadratic inequalities.

$$y < -x^2 + 3 \quad \text{Inequality 1}$$

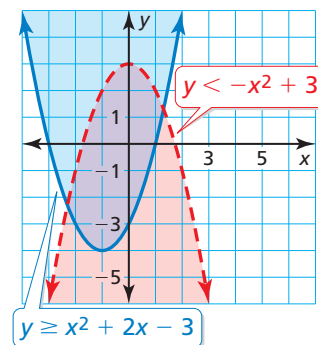
$$y \geq x^2 + 2x - 3 \quad \text{Inequality 2}$$

SOLUTION

Step 1 Graph $y < -x^2 + 3$. The graph is the red region inside (but not including) the parabola $y = -x^2 + 3$.

Step 2 Graph $y \geq x^2 + 2x - 3$. The graph is the blue region inside and including the parabola $y = x^2 + 2x - 3$.

Step 3 Identify the purple region where the two graphs overlap. This region is the graph of the system.



Check

Check that a point in the solution region, such as $(0, 0)$, is a solution of the system.

$$y < -x^2 + 3$$

$$0 \stackrel{?}{<} -0^2 + 3$$

$$0 < 3 \quad \checkmark$$

$$y \geq x^2 + 2x - 3$$

$$0 \stackrel{?}{\geq} 0^2 + 2(0) - 3$$

$$0 \geq -3 \quad \checkmark$$

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Graph the inequality.

1. $y \geq x^2 + 2x - 8$ 2. $y \leq 2x^2 - x - 1$ 3. $y > -x^2 + 2x + 4$

4. Graph the system of inequalities consisting of $y \leq -x^2$ and $y > x^2 - 3$.

Solving Quadratic Inequalities in One Variable

A **quadratic inequality in one variable** can be written in one of the following forms, where a , b , and c are real numbers and $a \neq 0$.

$$ax^2 + bx + c < 0 \quad ax^2 + bx + c > 0 \quad ax^2 + bx + c \leq 0 \quad ax^2 + bx + c \geq 0$$

You can solve quadratic inequalities using algebraic methods or graphs.

EXAMPLE 4 Solving a Quadratic Inequality Algebraically

Solve $x^2 - 3x - 4 < 0$ algebraically.

SOLUTION

First, write and solve the equation obtained by replacing $<$ with $=$.

$$x^2 - 3x - 4 = 0$$

Write the related equation.

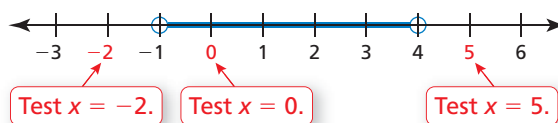
$$(x - 4)(x + 1) = 0$$

Factor.

$$x = 4 \quad \text{or} \quad x = -1$$

Zero-Product Property

The numbers -1 and 4 are the *critical values* of the original inequality. Plot -1 and 4 on a number line, using open dots because the values do not satisfy the inequality. The critical x -values partition the number line into three intervals. Test an x -value in each interval to determine whether it satisfies the inequality.



$$(-2)^2 - 3(-2) - 4 = 6 \not< 0 \quad 0^2 - 3(0) - 4 = -4 < 0 \quad 5^2 - 3(5) - 4 = 6 \not< 0$$

► So, the solution is $-1 < x < 4$.

Another way to solve $ax^2 + bx + c < 0$ is to first graph the related function $y = ax^2 + bx + c$. Then, because the inequality symbol is $<$, identify the x -values for which the graph lies *below* the x -axis. You can use a similar procedure to solve quadratic inequalities that involve \leq , $>$, or \geq .

EXAMPLE 5 Solving a Quadratic Inequality by Graphing

Solve $3x^2 - x - 5 \geq 0$ by graphing.

SOLUTION

The solution consists of the x -values for which the graph of $y = 3x^2 - x - 5$ lies on or above the x -axis. Find the x -intercepts of the graph by letting $y = 0$ and using the Quadratic Formula to solve $0 = 3x^2 - x - 5$ for x .

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-5)}}{2(3)}$$

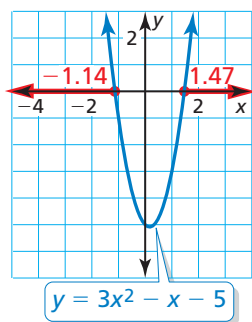
$$a = 3, b = -1, c = -5$$

$$x = \frac{1 \pm \sqrt{61}}{6}$$

Simplify.

The solutions are $x \approx -1.14$ and $x \approx 1.47$. Sketch a parabola that opens up and has -1.14 and 1.47 as x -intercepts. The graph lies on or above the x -axis to the left of (and including) $x = -1.14$ and to the right of (and including) $x = 1.47$.

► The solution of the inequality is approximately $x \leq -1.14$ or $x \geq 1.47$.



EXAMPLE 6 Modeling with Mathematics

A rectangular parking lot must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the parking lot.

SOLUTION

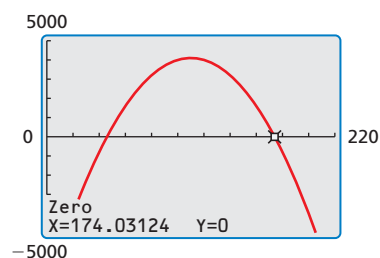
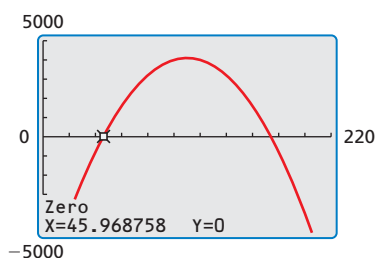
- 1. Understand the Problem** You are given the perimeter and the minimum area of a parking lot. You are asked to determine the possible lengths of the parking lot.
- 2. Make a Plan** Use the perimeter and area formulas to write a quadratic inequality describing the possible lengths of the parking lot. Then solve the inequality.
- 3. Solve the Problem** Let ℓ represent the length (in feet) and let w represent the width (in feet) of the parking lot.

$$\begin{aligned} \text{Perimeter} &= 440 & \text{Area} &\geq 8000 \\ 2\ell + 2w &= 440 & \ell w &\geq 8000 \end{aligned}$$

Solve the perimeter equation for w to obtain $w = 220 - \ell$. Substitute this into the area inequality to obtain a quadratic inequality in one variable.

$$\begin{aligned} \ell w &\geq 8000 && \text{Write the area inequality.} \\ \ell(220 - \ell) &\geq 8000 && \text{Substitute } 220 - \ell \text{ for } w. \\ 220\ell - \ell^2 &\geq 8000 && \text{Distributive Property} \\ -\ell^2 + 220\ell - 8000 &\geq 0 && \text{Write in standard form.} \end{aligned}$$

Use a graphing calculator to find the ℓ -intercepts of $y = -\ell^2 + 220\ell - 8000$.



The ℓ -intercepts are $\ell \approx 45.97$ and $\ell \approx 174.03$. The solution consists of the ℓ -values for which the graph lies on or above the ℓ -axis. The graph lies on or above the ℓ -axis when $45.97 \leq \ell \leq 174.03$.

► So, the approximate length of the parking lot is at least 46 feet and at most 174 feet.

- 4. Look Back** Choose a length in the solution region, such as $\ell = 100$, and find the width. Then check that the dimensions satisfy the original area inequality.

$$\begin{aligned} 2\ell + 2w &= 440 & \ell w &\geq 8000 \\ 2(100) + 2w &= 440 & 100(110) &\stackrel{?}{\geq} 8000 \\ w &= 110 & 11,000 &\geq 8000 \quad \checkmark \end{aligned}$$

ANOTHER WAY

You can graph each side of $220\ell - \ell^2 = 8000$ and use the intersection points to determine when $220\ell - \ell^2$ is greater than or equal to 8000.

USING TECHNOLOGY

Variables displayed when using technology may not match the variables used in applications. In the graphs shown, the length ℓ corresponds to the independent variable x and the area A corresponds to the dependent variable y .

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Solve the inequality.

5. $2x^2 + 3x \leq 2$ 6. $-3x^2 - 4x + 1 < 0$ 7. $2x^2 + 2 > -5x$

- 8. WHAT IF?** In Example 6, the area must be at least 8500 square feet. Describe the possible lengths of the parking lot.

3.6 Exercises

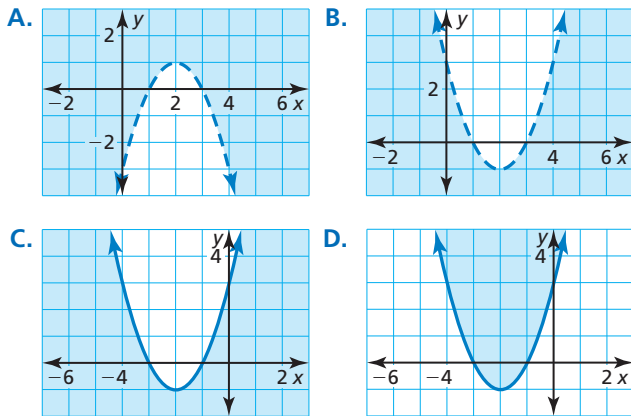
Vocabulary and Core Concept Check

- WRITING** Compare the graph of a quadratic inequality in one variable to the graph of a quadratic inequality in two variables.
- WRITING** Explain how to solve $x^2 + 6x - 8 < 0$ using algebraic methods and using graphs.

Monitoring Progress and Modeling with Mathematics

In Exercises 3–6, match the inequality with its graph. Explain your reasoning.

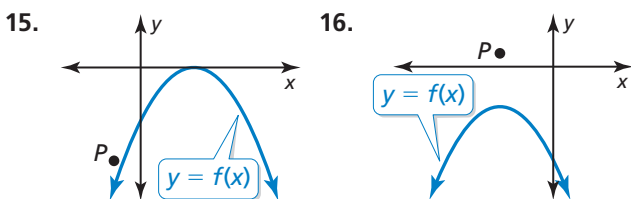
- | | |
|--------------------------|--------------------------|
| 3. $y \leq x^2 + 4x + 3$ | 4. $y > -x^2 + 4x - 3$ |
| 5. $y < x^2 - 4x + 3$ | 6. $y \geq x^2 + 4x + 3$ |



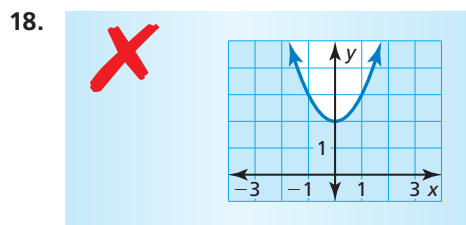
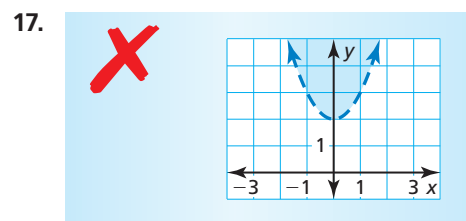
In Exercises 7–14, graph the inequality. (See Example 1.)

- | | |
|--------------------------|---|
| 7. $y < -x^2$ | 8. $y \geq 4x^2$ |
| 9. $y > x^2 - 9$ | 10. $y < x^2 + 5$ |
| 11. $y \leq x^2 + 5x$ | 12. $y \geq -2x^2 + 9x - 4$ |
| 13. $y > 2(x + 3)^2 - 1$ | 14. $y \leq \left(x - \frac{1}{2}\right)^2 + \frac{5}{2}$ |

ANALYZING RELATIONSHIPS In Exercises 15 and 16, use the graph to write an inequality in terms of $f(x)$ so point P is a solution.



ERROR ANALYSIS In Exercises 17 and 18, describe and correct the error in graphing $y \geq x^2 + 2$.



- MODELING WITH MATHEMATICS** A hardwood shelf in a wooden bookcase can safely support a weight W (in pounds) provided $W \leq 115x^2$, where x is the thickness (in inches) of the shelf. Graph the inequality and interpret the solution. (See Example 2.)
- MODELING WITH MATHEMATICS** A wire rope can safely support a weight W (in pounds) provided $W \leq 8000d^2$, where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.

In Exercises 21–26, graph the system of quadratic inequalities. (See Example 3.)

- | | |
|---|--|
| 21. $y \geq 2x^2$
$y < -x^2 + 1$ | 22. $y > -5x^2$
$y > 3x^2 - 2$ |
| 23. $y \leq -x^2 + 4x - 4$
$y < x^2 + 2x - 8$ | 24. $y \geq x^2 - 4$
$y \leq -2x^2 + 7x + 4$ |
| 25. $y \geq 2x^2 + x - 5$
$y < -x^2 + 5x + 10$ | 26. $y \geq x^2 - 3x - 6$
$y \geq x^2 + 7x + 6$ |

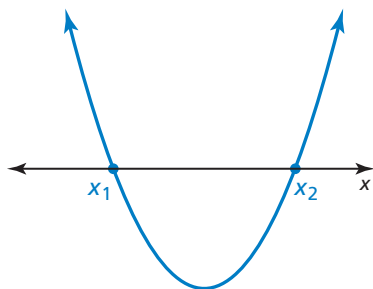
In Exercises 27–34, solve the inequality algebraically.
(See Example 4.)

27. $4x^2 < 25$ 28. $x^2 + 10x + 9 < 0$
 29. $x^2 - 11x \geq -28$ 30. $3x^2 - 13x > -10$
 31. $2x^2 - 5x - 3 \leq 0$ 32. $4x^2 + 8x - 21 \geq 0$
 33. $\frac{1}{2}x^2 - x > 4$ 34. $-\frac{1}{2}x^2 + 4x \leq 1$

In Exercises 35–42, solve the inequality by graphing.
(See Example 5.)

35. $x^2 - 3x + 1 < 0$ 36. $x^2 - 4x + 2 > 0$
 37. $x^2 + 8x > -7$ 38. $x^2 + 6x < -3$
 39. $3x^2 - 8 \leq -2x$ 40. $3x^2 + 5x - 3 < 1$
 41. $\frac{1}{3}x^2 + 2x \geq 2$ 42. $\frac{3}{4}x^2 + 4x \geq 3$

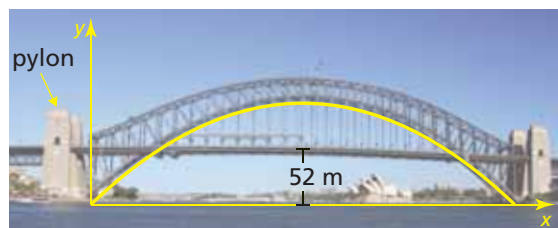
43. **DRAWING CONCLUSIONS** Consider the graph of the function $f(x) = ax^2 + bx + c$.



- What are the solutions of $ax^2 + bx + c < 0$?
 - What are the solutions of $ax^2 + bx + c > 0$?
 - The graph of g represents a reflection in the x -axis of the graph of f . For which values of x is $g(x)$ positive?
44. **MODELING WITH MATHEMATICS** A rectangular fountain display has a perimeter of 400 feet and an area of at least 9100 feet. Describe the possible widths of the fountain. (See Example 6.)



45. **MODELING WITH MATHEMATICS** The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by $y = -0.00211x^2 + 1.06x$, where x is the distance (in meters) from the left pylons and y is the height (in meters) of the arch above the water. For what distances x is the arch above the road?



46. **PROBLEM SOLVING** The number T of teams that have participated in a robot-building competition for high-school students over a recent period of time x (in years) can be modeled by

$$T(x) = 17.155x^2 + 193.68x + 235.81, 0 \leq x \leq 6.$$

After how many years is the number of teams greater than 1000? Justify your answer.



47. **PROBLEM SOLVING** A study found that a driver's reaction time $A(x)$ to audio stimuli and his or her reaction time $V(x)$ to visual stimuli (both in milliseconds) can be modeled by

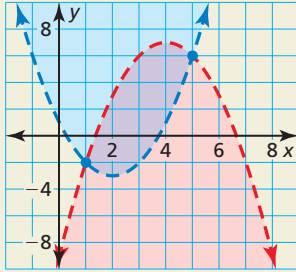
$$A(x) = 0.0051x^2 - 0.319x + 15, 16 \leq x \leq 70$$

$$V(x) = 0.005x^2 - 0.23x + 22, 16 \leq x \leq 70$$

where x is the age (in years) of the driver.

- Write an inequality that you can use to find the x -values for which $A(x)$ is less than $V(x)$.
- Use a graphing calculator to solve the inequality $A(x) < V(x)$. Describe how you used the domain $16 \leq x \leq 70$ to determine a reasonable solution.
- Based on your results from parts (a) and (b), do you think a driver would react more quickly to a traffic light changing from green to yellow or to the siren of an approaching ambulance? Explain.

48. **HOW DO YOU SEE IT?** The graph shows a system of quadratic inequalities.



- Identify two solutions of the system.
 - Are the points $(1, -2)$ and $(5, 6)$ solutions of the system? Explain.
 - Is it possible to change the inequality symbol(s) so that one, but not both of the points, is a solution of the system? Explain.
49. **MODELING WITH MATHEMATICS** The length L (in millimeters) of the larvae of the black porgy fish can be modeled by

$$L(x) = 0.00170x^2 + 0.145x + 2.35, 0 \leq x \leq 40$$

where x is the age (in days) of the larvae. Write and solve an inequality to find at what ages a larva's length tends to be greater than 10 millimeters. Explain how the given domain affects the solution.

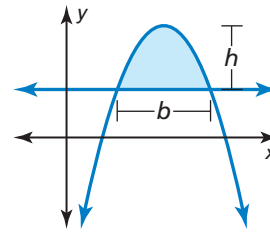


50. **MAKING AN ARGUMENT** You claim the system of inequalities below, where a and b are real numbers, has no solution. Your friend claims the system will always have at least one solution. Who is correct? Explain.

$$y < (x + a)^2$$

$$y < (x + b)^2$$

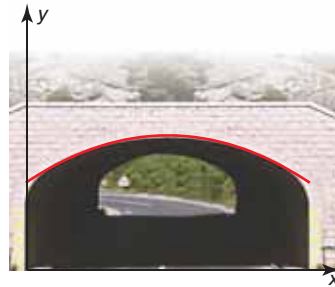
51. **MATHEMATICAL CONNECTIONS** The area A of the region bounded by a parabola and a horizontal line can be modeled by $A = \frac{2}{3}bh$, where b and h are as defined in the diagram. Find the area of the region determined by each pair of inequalities.



- $y \leq -x^2 + 4x$
 $y \geq 0$
- $y \geq x^2 - 4x - 5$
 $y \leq 7$

52. **THOUGHT PROVOKING** Draw a company logo that is created by the intersection of two quadratic inequalities. Justify your answer.

53. **REASONING** A truck that is 11 feet tall and 7 feet wide is traveling under an arch. The arch can be modeled by $y = -0.0625x^2 + 1.25x + 5.75$, where x and y are measured in feet.



- Will the truck fit under the arch? Explain.
- What is the maximum width that a truck 11 feet tall can have and still make it under the arch?
- What is the maximum height that a truck 7 feet wide can have and still make it under the arch?

Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Graph the function. Label the x -intercept(s) and the y -intercept. (Section 2.2)

54. $f(x) = (x + 7)(x - 9)$

55. $g(x) = (x - 2)^2 - 4$

56. $h(x) = -x^2 + 5x - 6$

Find the minimum value or maximum value of the function. Then describe where the function is increasing and decreasing. (Section 2.2)

57. $f(x) = -x^2 - 6x - 10$

58. $h(x) = \frac{1}{2}(x + 2)^2 - 1$

59. $f(x) = -(x - 3)(x + 7)$

60. $h(x) = x^2 + 3x - 18$