

2.4 Modeling with Quadratic Functions



Learning Standards
HSA-CED.A.2
HSF-IF.B.6
HSF-BF.A.1a
HSS-ID.B.6a

Essential Question How can you use a quadratic function to model a real-life situation?

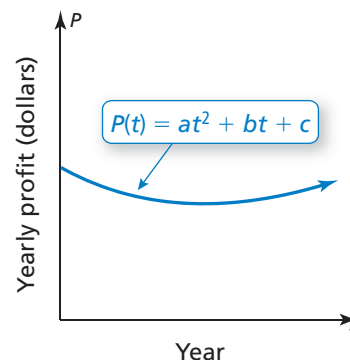
EXPLORATION 1 Modeling with a Quadratic Function

Work with a partner. The graph shows a quadratic function of the form

$$P(t) = at^2 + bt + c$$

which approximates the yearly profits for a company, where $P(t)$ is the profit in year t .

- Is the value of a positive, negative, or zero? Explain.
- Write an expression in terms of a and b that represents the year t when the company made the least profit.
- The company made the same yearly profits in 2004 and 2012. Estimate the year in which the company made the least profit.
- Assume that the model is still valid today. Are the yearly profits currently increasing, decreasing, or constant? Explain.

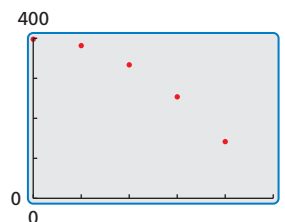


EXPLORATION 2 Modeling with a Graphing Calculator

Work with a partner. The table shows the heights h (in feet) of a wrench t seconds after it has been dropped from a building under construction.

Time, t	0	1	2	3	4
Height, h	400	384	336	256	144

- Use a graphing calculator to create a scatter plot of the data, as shown at the right. Explain why the data appear to fit a quadratic model.
- Use the *quadratic regression* feature to find a quadratic model for the data.
- Graph the quadratic function on the same screen as the scatter plot to verify that it fits the data.
- When does the wrench hit the ground? Explain.



MODELING WITH MATHEMATICS

To be proficient in math, you need to routinely interpret your results in the context of the situation.

Communicate Your Answer

- How can you use a quadratic function to model a real-life situation?
- Use the Internet or some other reference to find examples of real-life situations that can be modeled by quadratic functions.

2.4 Lesson

Core Vocabulary

Previous

average rate of change
system of three linear
equations

What You Will Learn

- ▶ Write equations of quadratic functions using vertices, points, and x-intercepts.
- ▶ Write quadratic equations to model data sets.

Writing Quadratic Equations

Core Concept

Writing Quadratic Equations

Given a point and the vertex (h, k)

Use vertex form:

$$y = a(x - h)^2 + k$$

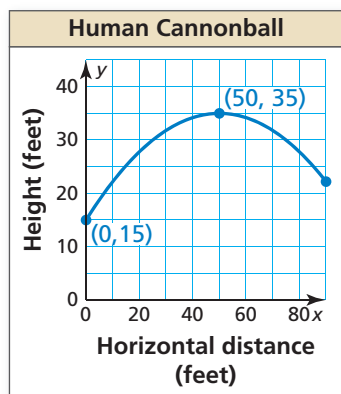
Given a point and x-intercepts p and q

Use intercept form:

$$y = a(x - p)(x - q)$$

Given three points

Write and solve a system of three equations in three variables.



EXAMPLE 1 Writing an Equation Using a Vertex and a Point

The graph shows the parabolic path of a performer who is shot out of a cannon, where y is the height (in feet) and x is the horizontal distance traveled (in feet). Write an equation of the parabola. The performer lands in a net 90 feet from the cannon. What is the height of the net?

SOLUTION

From the graph, you can see that the vertex (h, k) is $(50, 35)$ and the parabola passes through the point $(0, 15)$. Use the vertex and the point to solve for a in vertex form.

$$y = a(x - h)^2 + k$$

Vertex form

$$15 = a(0 - 50)^2 + 35$$

Substitute for $h, k, x,$ and y .

$$-20 = 2500a$$

Simplify.

$$-0.008 = a$$

Divide each side by 2500.

Because $a = -0.008$, $h = 50$, and $k = 35$, the path can be modeled by the equation $y = -0.008(x - 50)^2 + 35$, where $0 \leq x \leq 90$. Find the height when $x = 90$.

$$y = -0.008(90 - 50)^2 + 35$$

Substitute 90 for x .

$$= -0.008(1600) + 35$$

Simplify.

$$= 22.2$$

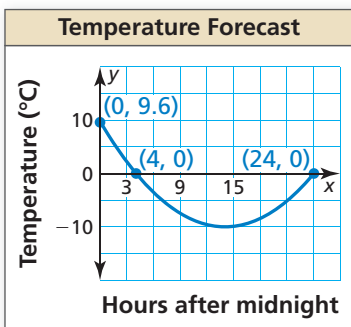
Simplify.

- ▶ So, the height of the net is about 22 feet.

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1. **WHAT IF?** The vertex of the parabola is $(50, 37.5)$. What is the height of the net?
2. Write an equation of the parabola that passes through the point $(-1, 2)$ and has vertex $(4, -9)$.

EXAMPLE 2 Writing an Equation Using a Point and x-Intercepts



REMEMBER

The average rate of change of a function f from x_1 to x_2 is the slope of the line connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

A meteorologist creates a parabola to predict the temperature tomorrow, where x is the number of hours after midnight and y is the temperature (in degrees Celsius).

- Write a function f that models the temperature over time. What is the coldest temperature?
- What is the average rate of change in temperature over the interval in which the temperature is decreasing? increasing? Compare the average rates of change.

SOLUTION

- The x -intercepts are 4 and 24 and the parabola passes through $(0, 9.6)$. Use the x -intercepts and the point to solve for a in intercept form.

$$y = a(x - p)(x - q)$$

Intercept form

$$9.6 = a(0 - 4)(0 - 24)$$

Substitute for p , q , x , and y .

$$9.6 = 96a$$

Simplify.

$$0.1 = a$$

Divide each side by 96.

Because $a = 0.1$, $p = 4$, and $q = 24$, the temperature over time can be modeled by $f(x) = 0.1(x - 4)(x - 24)$, where $0 \leq x \leq 24$. The coldest temperature is the minimum value. So, find $f(x)$ when $x = \frac{4 + 24}{2} = 14$.

$$f(14) = 0.1(14 - 4)(14 - 24)$$

Substitute 14 for x .

$$= -10$$

Simplify.

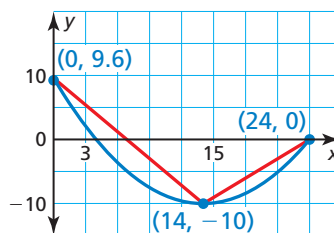
- So, the coldest temperature is -10°C at 14 hours after midnight, or 2 P.M.
- The parabola opens up and the axis of symmetry is $x = 14$. So, the function is decreasing over the interval $0 < x < 14$ and increasing over the interval $14 < x < 24$.

Average rate of change
over $0 < x < 14$:

Average rate of change
over $14 < x < 24$:

$$\frac{f(14) - f(0)}{14 - 0} = \frac{-10 - 9.6}{14} = -1.4$$

$$\frac{f(24) - f(14)}{24 - 14} = \frac{0 - (-10)}{10} = 1$$



- Because $|-1.4| > |1|$, the average rate at which the temperature decreases from midnight to 2 P.M. is greater than the average rate at which it increases from 2 P.M. to midnight.

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- WHAT IF?** The y -intercept is 4.8. How does this change your answers in parts (a) and (b)?
- Write an equation of the parabola that passes through the point $(2, 5)$ and has x -intercepts -2 and 4 .

Writing Equations to Model Data

When data have equally-spaced inputs, you can analyze patterns in the differences of the outputs to determine what type of function can be used to model the data. Linear data have constant *first differences*. Quadratic data have constant *second differences*. The first and second differences of $f(x) = x^2$ are shown below.

Equally-spaced x -values

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9

first differences: -5 -3 -1 1 3 5

second differences: 2 2 2 2 2

EXAMPLE 3 Writing a Quadratic Equation Using Three Points

Time, t	Height, h
10	26,900
15	29,025
20	30,600
25	31,625
30	32,100
35	32,025
40	31,400

NASA can create a weightless environment by flying a plane in parabolic paths. The table shows heights h (in feet) of a plane t seconds after starting the flight path. After about 20.8 seconds, passengers begin to experience a weightless environment. Write and evaluate a function to approximate the height at which this occurs.

SOLUTION

Step 1 The input values are equally spaced. So, analyze the differences in the outputs to determine what type of function you can use to model the data.

$h(10)$	$h(15)$	$h(20)$	$h(25)$	$h(30)$	$h(35)$	$h(40)$
26,900	29,025	30,600	31,625	32,100	32,025	31,400
	2125	1575	1025	475	-75	-625
		-550	-550	-550	-550	-550

Because the second differences are constant, you can model the data with a quadratic function.

Step 2 Write a quadratic function of the form $h(t) = at^2 + bt + c$ that models the data. Use any three points (t, h) from the table to write a system of equations.

Use (10, 26,900): $100a + 10b + c = 26,900$ Equation 1

Use (20, 30,600): $400a + 20b + c = 30,600$ Equation 2

Use (30, 32,100): $900a + 30b + c = 32,100$ Equation 3

Use the elimination method to solve the system.

Subtract Equation 1 from Equation 2.

→ $300a + 10b = 3700$

New Equation 1

Subtract Equation 1 from Equation 3.

→ $800a + 20b = 5200$

New Equation 2

$200a = -2200$

Subtract 2 times new Equation 1 from new Equation 2.

$a = -11$

Solve for a .

$b = 700$

Substitute into new Equation 1 to find b .

$c = 21,000$

Substitute into Equation 1 to find c .

The data can be modeled by the function $h(t) = -11t^2 + 700t + 21,000$.

Step 3 Evaluate the function when $t = 20.8$.

$$h(20.8) = -11(20.8)^2 + 700(20.8) + 21,000 = 30,800.96$$

▶ Passengers begin to experience a weightless environment at about 30,800 feet.

Real-life data that show a quadratic relationship usually do not have constant second differences because the data are not *exactly* quadratic. Relationships that are *approximately* quadratic have second differences that are relatively “close” in value. Many technology tools have a *quadratic regression* feature that you can use to find a quadratic function that best models a set of data.

EXAMPLE 4 Using Quadratic Regression

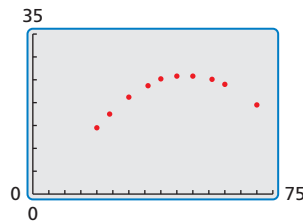
Miles per hour, x	Miles per gallon, y
20	14.5
24	17.5
30	21.2
36	23.7
40	25.2
45	25.8
50	25.8
56	25.1
60	24.0
70	19.5

The table shows fuel efficiencies of a vehicle at different speeds. Write a function that models the data. Use the model to approximate the optimal driving speed.

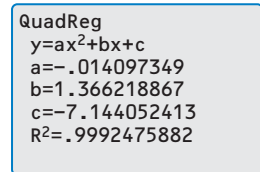
SOLUTION

Because the x -values are not equally spaced, you cannot analyze the differences in the outputs. Use a graphing calculator to find a function that models the data.

Step 1 Enter the data in a graphing calculator using two lists and create a scatter plot. The data show a quadratic relationship.



Step 2 Use the *quadratic regression* feature. A quadratic model that represents the data is $y = -0.014x^2 + 1.37x - 7.1$.

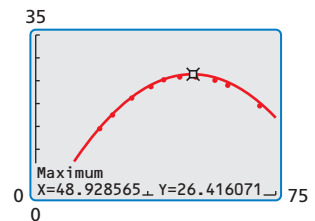


STUDY TIP

The *coefficient of determination* R^2 shows how well an equation fits a set of data. The closer R^2 is to 1, the better the fit.

Step 3 Graph the regression equation with the scatter plot.

In this context, the “optimal” driving speed is the speed at which the mileage per gallon is maximized. Using the *maximum* feature, you can see that the maximum mileage per gallon is about 26.4 miles per gallon when driving about 48.9 miles per hour.



► So, the optimal driving speed is about 49 miles per hour.

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- Write an equation of the parabola that passes through the points $(-1, 4)$, $(0, 1)$, and $(2, 7)$.
- The table shows the estimated profits y (in dollars) for a concert when the charge is x dollars per ticket. Write and evaluate a function to determine what the charge per ticket should be to maximize the profit.

Ticket price, x	2	5	8	11	14	17
Profit, y	2600	6500	8600	8900	7400	4100

- The table shows the results of an experiment testing the maximum weights y (in tons) supported by ice x inches thick. Write a function that models the data. How much weight can be supported by ice that is 22 inches thick?

Ice thickness, x	12	14	15	18	20	24	27
Maximum weight, y	3.4	7.6	10.0	18.3	25.0	40.6	54.3

2.4 Exercises

Vocabulary and Core Concept Check

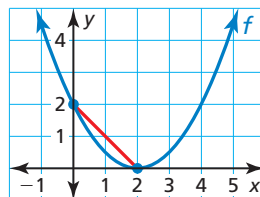
- WRITING** Explain when it is appropriate to use a quadratic model for a set of data.
- DIFFERENT WORDS, SAME QUESTION** Which is different? Find “both” answers.

What is the average rate of change over $0 \leq x \leq 2$?

What is the distance from $f(0)$ to $f(2)$?

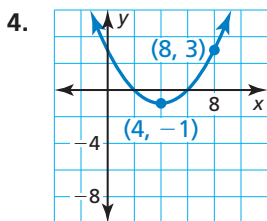
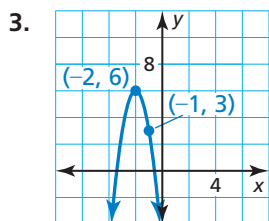
What is the slope of the line segment?

What is $\frac{f(2) - f(0)}{2 - 0}$?



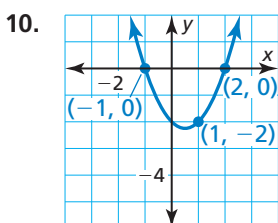
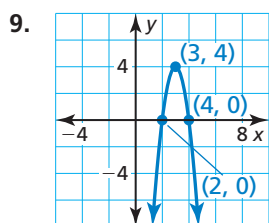
Monitoring Progress and Modeling with Mathematics

In Exercises 3–8, write an equation of the parabola in vertex form. (See Example 1.)



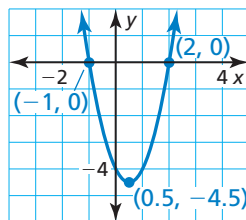
- passes through (13, 8) and has vertex (3, 2)
- passes through (-7, -15) and has vertex (-5, 9)
- passes through (0, -24) and has vertex (-6, -12)
- passes through (6, 35) and has vertex (-1, 14)

In Exercises 9–14, write an equation of the parabola in intercept form. (See Example 2.)



- x-intercepts of 12 and -6; passes through (14, 4)
- x-intercepts of 9 and 1; passes through (0, -18)
- x-intercepts of -16 and -2; passes through (-18, 72)
- x-intercepts of -7 and -3; passes through (-2, 0.05)

- WRITING** Explain when to use intercept form and when to use vertex form when writing an equation of a parabola.
- ANALYZING EQUATIONS** Which of the following equations represent the parabola?

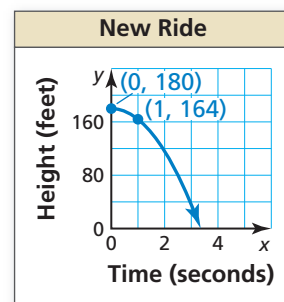
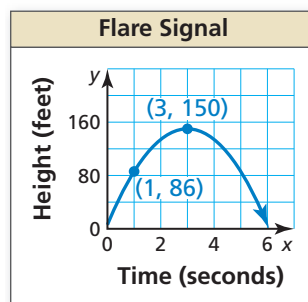


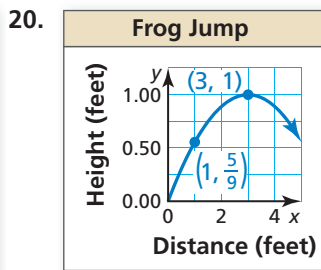
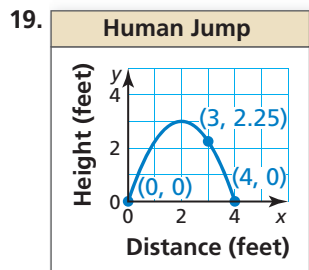
- $y = 2(x - 2)(x + 1)$
- $y = 2(x + 0.5)^2 - 4.5$
- $y = 2(x - 0.5)^2 - 4.5$
- $y = 2(x + 2)(x - 1)$

In Exercises 17–20, write an equation of the parabola in vertex form or intercept form.

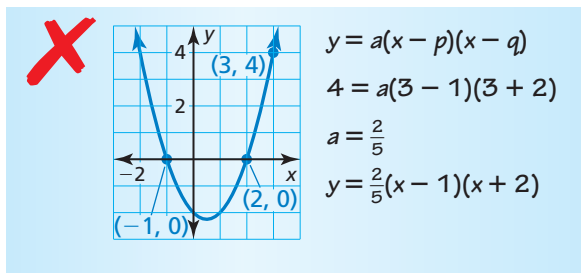
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18.

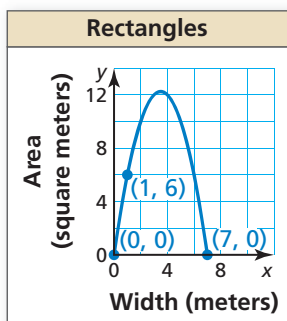




21. **ERROR ANALYSIS** Describe and correct the error in writing an equation of the parabola.



22. **MATHEMATICAL CONNECTIONS** The area of a rectangle is modeled by the graph where y is the area (in square meters) and x is the width (in meters). Write an equation of the parabola. Find the dimensions and corresponding area of one possible rectangle. What dimensions result in the maximum area?



23. **MODELING WITH MATHEMATICS** Every rope has a safe working load. A rope should not be used to lift a weight greater than its safe working load. The table shows the safe working loads S (in pounds) for ropes with circumference C (in inches). Write an equation for the safe working load for a rope. Find the safe working load for a rope that has a circumference of 10 inches. (See Example 3.)

Circumference, C	0	1	2	3
Safe working load, S	0	180	720	1620

24. **MODELING WITH MATHEMATICS** A baseball is thrown up in the air. The table shows the heights y (in feet) of the baseball after x seconds. Write an equation for the path of the baseball. Find the height of the baseball after 5 seconds.

Time, x	0	2	4	6
Baseball height, y	6	22	22	6

25. **COMPARING METHODS** You use a system with three variables to find the equation of a parabola that passes through the points $(-8, 0)$, $(2, -20)$, and $(1, 0)$. Your friend uses intercept form to find the equation. Whose method is easier? Justify your answer.
26. **MODELING WITH MATHEMATICS** The table shows the distances y a motorcyclist is from home after x hours.

Time (hours), x	0	1	2	3
Distance (miles), y	0	45	90	135

- a. Determine what type of function you can use to model the data. Explain your reasoning.
- b. Write and evaluate a function to determine the distance the motorcyclist is from home after 6 hours.
27. **USING TOOLS** The table shows the heights h (in feet) of a sponge t seconds after it was dropped by a window cleaner on top of a skyscraper. (See Example 4.)

Time, t	0	1	1.5	2.5	3
Height, h	280	264	244	180	136

- a. Use a graphing calculator to create a scatter plot. Which better represents the data, a line or a parabola? Explain.
- b. Use the *regression* feature of your calculator to find the model that best fits the data.
- c. Use the model in part (b) to predict when the sponge will hit the ground.
- d. Identify and interpret the domain and range in this situation.
28. **MAKING AN ARGUMENT** Your friend states that quadratic functions with the same x -intercepts have the same equations, vertex, and axis of symmetry. Is your friend correct? Explain your reasoning.

In Exercises 29–32, analyze the differences in the outputs to determine whether the data are *linear*, *quadratic*, or *neither*. Explain. If linear or quadratic, write an equation that fits the data.

29.

Price decrease (dollars), x	0	5	10	15	20
Revenue (\$1000s), y	470	630	690	650	510

30.

Time (hours), x	0	1	2	3	4
Height (feet), y	40	42	44	46	48

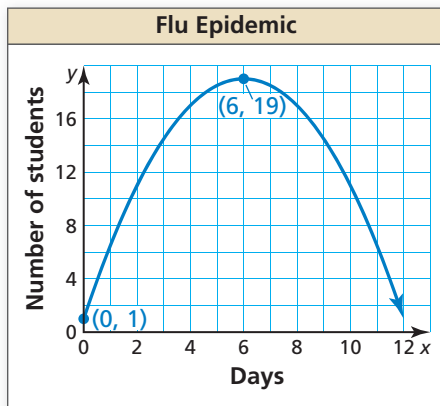
31.

Time (hours), x	1	2	3	4	5
Population (hundreds), y	2	4	8	16	32

32.

Time (days), x	0	1	2	3	4
Height (feet), y	320	303	254	173	60

33. **PROBLEM SOLVING** The graph shows the number y of students absent from school due to the flu each day x .



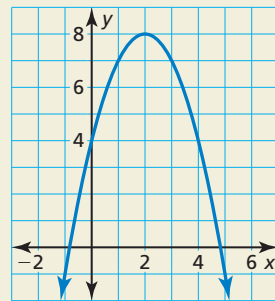
- Interpret the meaning of the vertex in this situation.
- Write an equation for the parabola to predict the number of students absent on day 10.
- Compare the average rates of change in the students with the flu from 0 to 6 days and 6 to 11 days.

34. **THOUGHT PROVOKING** Describe a real-life situation that can be modeled by a quadratic equation. Justify your answer.

35. **PROBLEM SOLVING** The table shows the heights y of a competitive water-skier x seconds after jumping off a ramp. Write a function that models the height of the water-skier over time. When is the water-skier 5 feet above the water? How long is the skier in the air?

Time (seconds), x	0	0.25	0.75	1	1.1
Height (feet), y	22	22.5	17.5	12	9.24

36. **HOW DO YOU SEE IT?** Use the graph to determine whether the average rate of change over each interval is *positive*, *negative*, or *zero*.



- $0 \leq x \leq 2$
- $2 \leq x \leq 5$
- $2 \leq x \leq 4$
- $0 \leq x \leq 4$

37. **REPEATED REASONING** The table shows the number of tiles in each figure. Verify that the data show a quadratic relationship. Predict the number of tiles in the 12th figure.

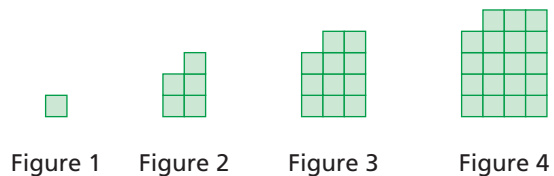


Figure	1	2	3	4
Number of Tiles	1	5	11	19

Maintaining Mathematical Proficiency Reviewing what you learned in previous grades and lessons

Factor the trinomial. (*Skills Review Handbook*)

38. $x^2 + 4x + 3$ 39. $x^2 - 3x + 2$ 40. $3x^2 - 15x + 12$ 41. $5x^2 + 5x - 30$